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STRONGLY REGULAR GRAPHS DEFINED BY SPREADS

BY

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ABSTRACT

Spreads of finite symplectic, orthogonal and unitary vector spaces are used to construct new strongly regular graphs having the same parameters as the perpendicularity graphs of the underlying vector spaces. Some of the graphs are related to partial geometries, while others produce interesting symmetric designs.

1. Introduction

Let V be a vector space over $GF(q)$, equipped with a symplectic, orthogonal or unitary geometry. A *spread* of V is a family Σ of maximal totally isotropic or singular subspaces which partitions the set $\mathbf{P}$ of totally isotropic or singular points. Let $m = \dim M$ for $M \in \Sigma$, and assume that $m \geqq 3$. Using Σ, we will construct a strongly regular graph $\mathbf{G}(\Sigma)$. The parameters of $\mathbf{G}(\Sigma)$ are the same as those of the classical strongly regular graph $\mathbf{G}(V) = (\mathbf{P}, \perp)$, where $\perp$ denotes the relation "distinct but perpendicular"; however, the strongly regular graphs $\mathbf{G}(\Sigma)$ and $\mathbf{G}(V)$ need not be isomorphic.

When V has type $\Omega^+(2m, 2)$ or $\Omega^+(2m, 3)$, $\mathbf{G}(\Sigma)$ is the line-graph of the partial geometry found by DeClerck, Dye and Thas [2, 14]. When V has type $\mathrm{Sp}(2m, q)$, $\mathbf{G}(\Sigma)$ determines a symmetric design having the same parameters as $PG(2m-1, q)$. Our goal is not just to define these graphs, partial geometries and designs: we will also indicate some of their properties.

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2. Spreads

We begin by summarizing some of the properties of V which will be needed later. Proofs can be found in Dieudonné [4].

If V is as in §1 then it is equipped with a form (bilinear, quadratic or hermitian) and a notion of perpendicularity. A subspace on which the form vanishes is called *totally isotropic* (for symplectic or unitary V) or *totally singular* (for orthogonal V); such a subspace is perpendicular to itself, but the converse is false for orthogonal geometries in characteristic 2. If X is any subspace then $\dim X^\perp = \dim V - \dim X$; in particular, if $X \leqq X^\perp$ then $\dim X \leqq \frac{1}{2}\dim V$. All maximal totally isotropic or singular subspaces have the same dimension $m \leqq \frac{1}{2}\dim V$.

Throughout this paper, V, Σ, $\mathbf{P}$, $\mathbf{G}(V)$ and m will have the same meaning as in §1. $|\mathbf{P}| = (q^{m+\varepsilon}+1)(q^m-1)/(q-1)$, where m and ε are related to V as in the following table:

Type of V	$\mathrm{Sp}(2m,q)$	$\Omega(2m+1,q)$	$\Omega^+(2m,q)$	$\Omega^-(2m+2,q)$	$U(2m,q^{1/2})$	$U(2m+1,q^{1/2})$
ε	0	0	-1	1	$-\frac{1}{2}$	$\frac{1}{2}$

In [4] the orthogonal geometries of type $\Omega^+(2m,q)$ and $\Omega^-(2m+2,q)$ are those having maximal and non-maximal index, respectively.

By definition, $|\mathbf{P}| = |\Sigma|(q^m-1)/(q-1)$. Consequently, $|\Sigma| = q^{m+\varepsilon}+1$.

We are only interested in the case $m \geqq 3$. The only spaces in which spreads are *known* to exist are then as follows (Dillon [5]; Dye [6]; Thas [13]; Kantor [9, 10]):

$\mathrm{Sp}(2m,q)$. All m,q.

$\Omega(2m+1,q)$. All m if q is even; $m=3$, $q=0$ or 2 (mod 3).

$\Omega^+(2m,q)$. All even m if q is even; $m=4$, $q=0$ or 2 (mod 3).

$\Omega^-(2m+2,q)$. All m if q is even.

Examples of spreads in some of these cases will be given later. No examples exist in $\Omega^+(2m,q)$ spaces if m is odd. Nothing is known about existence or nonexistence for unitary spaces.

3. The graphs $\mathbf{G}(\Sigma)$

Let V, $\mathbf{P}$ and Σ be as before.

Let Ω be the set of all hyperplanes of members of Σ. If $X \in \Omega$, let $\Sigma(X)$ denote that member of Σ containing X.

Write $X \sim Y \Leftrightarrow X \cap Y^{\perp} \neq 0$, where $X, Y \in \Omega$ and $X \neq Y$.
Set $\mathbf{G}(\Sigma) = (\Omega, \sim)$.

THEOREM 3.1. *$\mathbf{G}(V)$ and $\mathbf{G}(\Sigma)$ are strongly regular graphs having the same parameters.*

PROOF. Clearly, $|\mathbf{P}| = |\Sigma|(q^m - 1)/(q-1) = |\Omega|$. It is well-known that $\mathbf{G}(V)$ is strongly regular; its parameters are

$$|\mathbf{P}|, k = q(q^{m-1}-1)(q^{m+\varepsilon-1}+1)/(q-1),$$

$$\lambda = q - 1 + q^2(q^{m-2}-1)(q^{m+\varepsilon-2}+1)/(q-1), \quad \mu = (q^{m-1}-1)(q^{m+\varepsilon-1}+1)/(q-1).$$

We will check that $\mathbf{G}(\Sigma)$ is a strongly regular graph with these parameters, proceeding in several steps. The letters X, Y and Z will always denote members of Ω, while x and y will belong to $\mathbf{P}$.

(1) If $\Sigma(X) \neq M \in \Sigma$, then $X^{\perp} \cap M \in \mathbf{P}$. For, $\dim X^{\perp} = \dim V - \dim X = \dim V - (m-1)$, so that $\dim(X^{\perp} \cap M) = (\dim V - (m-1)) + m - \dim\langle X^{\perp}, M\rangle \geqq 1$. The maximality of m now shows that $\dim(X^{\perp} \cap M) = 1$.

(2) If $X \sim Y$ then $Y \sim X$. For, this is clear if $\Sigma(X) = \Sigma(Y)$, so assume that $\Sigma(X) \neq \Sigma(Y)$. Set $x = X \cap Y^{\perp}$ and $y = X^{\perp} \cap \Sigma(Y)$. Then $y \in X^{\perp} < x^{\perp}$, so $y \in X^{\perp} \cap (x^{\perp} \cap \Sigma(Y)) = X^{\perp} \cap Y$.

(3) Let $X \in \Omega$. Clearly, $X \sim Y$ whenever $X \neq Y < \Sigma(X)$. Let $M \in \Sigma - \{\Sigma(X)\}$, and set $y = X^{\perp} \cap M$. Then $X \sim Y$ whenever $y \in Y < M$. Thus, $\mathbf{G}(\Sigma)$ has valence $(q^m - q)/(q-1) + q^{m+\varepsilon} \cdot (q^{m-1}-1)/(q-1) = k$.

(4) Let $X \sim Y$ with $\Sigma(X) = \Sigma(Y)$. If $Z \neq X, Y$ and $Z < \Sigma(X)$ then $Z \sim X, Y$. This accounts for $(q^m - 1)/(q-1) - 2$ members of Ω. We now search for all $Z \sim X, Y$ with $\Sigma(Z) \neq \Sigma(X)$.

Let $M \in \Sigma - \{\Sigma(X)\}$. Form $X^{\perp} \cap M = x$ and $Y^{\perp} \cap M = y$. If $Z < M$, then $Z \sim X, Y$ precisely when $x, y \in Z$. This accounts for $q^{m+\varepsilon}(q^{m-2}-1)/(q-1)$ members of Ω.

Thus, there are $q^{m+\varepsilon}(q^{m-2}-1)/(q-1) + (q^m-1)/(q-1) - 2 = \lambda$ subspaces $Z \in \Omega$ such that $Z \sim X, Y$.

(5) Let $X \sim Y$ with $\Sigma(X) \neq \Sigma(Y)$. If $Z \neq X$ and $Y^{\perp} \cap \Sigma(X) \in Z < \Sigma(X)$, then $Z \sim X, Y$. Reversing the roles of X and Y, we obtain $2(q^{m-1} - q)/(q-1)$ such subspaces $Z \sim X, Y$.

Next, let $M \in \Sigma - \{\Sigma(X), \Sigma(Y)\}$, and set $x = X^{\perp} \cap M$ and $y = Y^{\perp} \cap M$. If $Z < M$, then $Z \sim X, Y$ precisely when $x, y \in Z$.

If $x = y$ then $x \in \langle X, Y\rangle^{\perp}$. Then $\langle x, X, X^{\perp} \cap Y\rangle$ is contained in $\mathbf{P}$ and has dimension $\geqq m$, so that $x \in \langle X, Y\rangle^{\perp} \cap \langle X, X^{\perp} \cap Y\rangle = \langle X^{\perp} \cap Y, X \cap Y^{\perp}\rangle$. Conversely, each of the $q-1$ members of $\Sigma-\{\Sigma(X),\Sigma(Y)\}$ meeting the latter line produces an instance of $x=y$. Consequently, exactly $(q-1)\cdot(q^{m-1}-1)/(q-1)$ subspaces $Z\sim X, Y$ arise in this manner. Similarly, if $x\neq y$ we obtain $(q^{m+\varepsilon}-q)\cdot(q^{m-2}-1)/(q-1)$ subspaces Z. Thus, the number of $Z\sim X, Y$ is

$$2(q^{m-1}-q)/(q-1)+(q-1)(q^{m-1}-1)/(q-1)+(q^{m+\varepsilon}-q)(q^{m-2}-1)/(q-1)=\lambda.$$

(6) Let $X\not\sim Y$. If $Y^{\perp}\cap\Sigma(X)<Z<\Sigma(X)$ then $Z\sim X, Y$. This produces $2(q^{m-1}-1)/(q-1)$ subspaces $Z\sim X, Y$ lying in $\Sigma(X)$ or $\Sigma(Y)$.

If $M\in\Sigma-\{\Sigma(X),\Sigma(Y)\}$, set $x=X^{\perp}\cap M$ and $y=Y^{\perp}\cap M$. If $Z<M$, then $Z\sim X, Y$ precisely when $x, y\in Z$.

Suppose that $x=y$. Then $x\in X^{\perp}\cap Y^{\perp}=\langle X, Y\rangle^{\perp}$. This is a nonsingular subspace of dimension $\dim V-2(m-1)$. Checking all cases, we find that it contains exactly $q^{\varepsilon+1}+1$ members of $\mathbf{P}$. This produces

$$(q^{\varepsilon+1}-1)\cdot(q^{m-1}-1)/(q-1)$$

subspaces Z.

Finally, if $x\neq y$ we obtain $(q^{m+\varepsilon}-q^{\varepsilon+1})\cdot(q^{m-2}-1)/(q-1)$ subspaces $Z\sim X, Y$. Since

$$\mu=2(q^{m-1}-1)/(q-1)+(q^{\varepsilon+1}-1)(q^{m-1}-1)/(q-1)$$
$$+(q^{m+\varepsilon}-q^{\varepsilon+1})(q^{m-2}-1)/(q-1),$$

this completes the proof of the theorem.

Definitions 3.2. (i) If $x\in M\in\Sigma$ then $M^*=\{Z\in\Omega\mid Z<M\}$ and $x^*=\{Z\in\Omega\mid x\in Z<M\}$.

(ii) $\Sigma^*=\{M^*\mid M\in\Sigma\}$.

(iii) Let $X\sim Y$. If $\Sigma(X)=\Sigma(Y)$, set $XY=\{Z\in\Omega\mid X\cap Y<Z<\Sigma(X)\}$; if $\Sigma(X)\neq\Sigma(Y)$, set

$$XY=\{Z\in\Omega\mid Z\cap\langle X\cap Y^{\perp}, X^{\perp}\cap Y\rangle\neq 0, Z^{\perp}\geqq\langle X\cap Y^{\perp}, X^{\perp}\cap Y\rangle\}.$$

In the latter case, every member of Σ meeting $\langle X\cap Y^{\perp}, X^{\perp}\cap Y\rangle$ nontrivially contains a unique member of XY (compare step (5) of the proof of (3.1)). Thus, $|XY|=q+1$ in any case.

(iv) If $X\sim Y$, $L(X, Y)=\{X, Y\}\cup\{Z\in\Omega\mid W\in\Omega-\{Z\}$ and $W\sim X, Y\Rightarrow W\sim Z\}$.

(v) Let $X\sim Y$. If $\Sigma(X)=\Sigma(Y)$, set $L_0(X, Y)=XY$; if $\Sigma(X)\neq\Sigma(Y)$, set $L_0(X, Y)=\{X, Y\}\cup\{Z\in\Omega\mid W\in\Omega-\{Z\},\ W\sim X, Y$, and $W^{\perp}>X^{\perp}\cap Y$ or $X\cap Y^{\perp}\Rightarrow W\sim Z\}$.

Evidently, XY, $L(X,Y)$ and $L_0(X,Y)$ are all versions of "lines". Only $L(X,Y)$ depends strictly upon the graph $\mathbf{G}(\Sigma)$. In computational situations, $L_0(X,Y)$ is easier to deal with than $L(X,Y)$.

These definitions are related, in view of the following simple lemmas.

LEMMA 3.3. (i) *Σ^* is a partition of Ω into maximal cliques.*

(ii) *If $\Sigma(X) \neq M \in \Sigma$ then $\{Z \in M^* \mid X \sim Z\} = x^*$ for a unique point x of M. Thus, the subsets x^* of M^* can be recovered from $\mathbf{G}(\Sigma)$, so that each clique M^* inherits from $\mathbf{G}(\Sigma)$ a natural structure as a projective space* $\mathrm{PG}(m-1,q)$.

(iii) *If $X \sim Y$ and $\Sigma(X) = \Sigma(Y)$, then $L(X,Y) = L_0(X,Y) = XY$.*

(iv) *If $X \sim Y$ then $L(X,Y) \subseteq L_0(X,Y) \subseteq XY$.*

LEMMA 3.4. *Assume that Σ^* contains every clique C of size $(q^m-1)/(q-1)$ such that $X, Y \in C$ and $X \neq Y$ imply that $L(X,Y) \subset C$ and $|L(X,Y)| = q+1$. Then* $\operatorname{Aut}\mathbf{G}(\Sigma)$ *is induced by the group of automorphisms of $\mathbf{G}(V)$ which send Σ to itself.*

PROOF. By (3.3i) and hypothesis, $\mathbf{G}(\Sigma)$ uniquely determines Σ^*. Then (3.3ii) can be used to recover $\mathbf{G}(V)$ from $\mathbf{G}(\Sigma)$: simply interchange the roles of $\mathbf{P}$, Σ and Ω, Σ^* in the construction described at the beginning of this section.

REMARKS. Aut $\mathbf{G}(V)$ is well-known (Dieudonné [4, ch. II, §3]): it consists of all invertible semilinear transformations of V which preserve the underlying form projectively.

It seems likely that the hypothesis of (3.4) holds whenever V does not have type $\Omega^+(8,q)$ (compare (4.1), (4.2)). In those cases we have computed, $|L(X,Y)| = 2$ whenever $X \sim Y$ and $\Sigma(X) \neq \Sigma(Y)$; in fact, except in the situation of the next lemma it appears that $|L_0(X,Y)| = 2$ when $\Sigma(X) \neq \Sigma(Y)$.

LEMMA 3.5. *Let V have type $\Omega^+(2m,q)$.*

(i) *If $x \in M \in \Sigma$, then x^* lies in exactly two maximal cliques: M^* and $C(x^*) = x^* \cup \{x^\perp \cap N \mid N \in \Sigma - \{M\}\}$.*

(ii) *If $X \sim Y$ then $|L_0(X,Y)| = q+1$.*

PROOF. (i) Let S be a clique containing x^*, and let $Y \in S - M^*$. If $X \in x^*$ then $Y^\perp \cap M \in X$, and hence $Y^\perp \cap M = x$. Thus, $S \subseteq C(x^*)$.

Let $Y, Z \in C(x^*) - x^*$, $Y \neq Z$. Then $V = \langle \Sigma(Y), \Sigma(X) \rangle$. Let $x \in \langle y, z \rangle$ with $y \in \Sigma(Y)$ and $z \in \Sigma(Z)$. Then $\langle y, z \rangle$ is totally singular since it has at least three singular points. Then $y \in x^\perp \cap \Sigma(Y) = Y$, while $x, z \in Z^\perp$, so that $y \in Y \cap Z^\perp$. Thus, $C(x^*)$ is a clique.

(ii) We may assume that $\Sigma(X) \neq \Sigma(Y)$. Set $x = X \cap Y^{\perp}$, and let $Z \in XY - \{X, Y\}$. If $W \neq X, Y$ and $W < x^{\perp}$, then $Z, W \in C(x^*)$ and hence $W \sim Z$. Thus, $XY = L_0(X, Y)$.

4. Are the graphs new?

This and the next two sections are concerned with special cases of the following

CONJECTURE 4.1. $\mathbf{G}(\Sigma) \cong \mathbf{G}(V')$ *for some symplectic, orthogonal or unitary space* V' *if, and only if,* V *has type* $\Omega^+(8, q)$.

If $\mathbf{G}(\Sigma) \cong \mathbf{G}(V')$ then $\mathbf{G}(V)$ and $\mathbf{G}(V')$ have the same parameters. Thus, V' can be taken to be V except perhaps if V has type $\mathrm{Sp}(2m, q)$ or $\Omega(2m+1, q)$ with q odd (see (4.6) for those cases).

PROPOSITION 4.2. *If* V *has type* $\Omega^+(8, q)$ *then* $\mathbf{G}(V) \cong \mathbf{G}(\Sigma)$.

PROOF. The totally singular 4-spaces of V fall into two classes, two 4-spaces lying in the same class if and only if their intersection has even dimension (Dieudonné [4, pp. 50, 65]). Each class contains $|\mathbf{P}|$ subspaces. If $X \in \Omega$, then X lies in exactly two totally singular 4-spaces: $\Sigma(X)$ and $\mathbf{M}(X)$, say. Since Σ lies in one class, $\mathbf{M} = \{\mathbf{M}(X) \mid X \in \Omega\}$ lies in the other. A class has size $|\mathbf{P}| = |\Omega| = |\mathbf{M}|$. Thus, $\mathbf{M}$ is an entire class.

If $X \sim Y$ and $\Sigma(X) = \Sigma(Y)$, then $\mathbf{M}(X) \cap \mathbf{M}(Y)$ is the line $X \cap Y$. If $X \sim Y$ and $\Sigma(X) \neq \Sigma(Y)$ then $\mathbf{M}(X) \cap \mathbf{M}(Y) = \langle X \cap Y^{\perp}, X^{\perp} \cap Y \rangle$. Consequently, a triality map (Tits [15]) sending $\mathbf{M}$ to $\mathbf{P}$ induces an isomorphism $\mathbf{G}(\Sigma) \cong \mathbf{G}(V)$.

DEFINITION. An *ovoid* of V is a set of $q^{m+\epsilon} + 1$ points in $\mathbf{P}$, no two of which are perpendicular.

If $\mathbf{O}$ is any set of pairwise nonperpendicular points of $\mathbf{P}$, by counting the pairs (x, E) with E a totally isotropic or singular m-space and $x \in \mathbf{O} \cap E$ we find that $|\mathbf{O}| \leqq q^{m+\epsilon} + 1$. Thus, ovoids have maximal size for such sets of points. Moreover $\mathbf{O}$ is an ovoid if and only if every totally isotropic or singular m-space of V meets $\mathbf{O}$. Consequently, if $\mathbf{O}$ is an ovoid and $y \in \mathbf{P} - \mathbf{O}$, then $\mathbf{O}_y = \{\langle x, y \rangle / y \mid x \in \mathbf{O} \cap y^{\perp}\}$ is an ovoid of $y^{\perp}/y$ (Thas [13]).

THEOREM 4.3. *V does not have an ovoid if V has type* (i) $\mathrm{Sp}(m, q)$ *or* (ii) $\Omega^+(2m, 2)$, $m > 4$.

PROOF. (i) This is due to Thas [13].

(ii) Let $\mathbf{O}$ be an ovoid. Since $\mathbf{O}_y$ is also an ovoid for $y \in \mathbf{P}-\mathbf{O}$, we may assume that $m=5$. Now $\mathbf{O}_y$ is an ovoid in an $\Omega^+(8,2)$-space, and hence is unique up to a change of coordinates (Dye [6], Thas [13]). Thus, $\mathbf{O}_y$ can be described as follows.

Take Z_2^9, with standard basis $e_1,\cdots,e_9$. Set $Q(e_i)=1$ and $(e_i,e_j)=1$ for $i\neq j$. This produces a quadratic form on Z_2^9, with radical spanned by $w=\Sigma_1^9 e_i$. We can identify $y^\perp/y$ with $\{\Sigma_1^9 \alpha_i e_i \mid \Sigma_1^9 \alpha_i=0\}$ and $\mathbf{O}_y$ with $\{\langle w+e_i\rangle \mid i=1,\cdots,9\}$. Note that $\langle w+e_i \mid i=1,\cdots,6\rangle^\perp=\langle e_7+e_8, e_8+e_9\rangle$ has no singular points.

Now count in two ways the number N of pairs (y,S) with $y\in\mathbf{P}-\mathbf{O}$, $S\subset\mathbf{O}\cap y^\perp$ and $|S|=6$. On the one hand,

$$N=|\mathbf{P}-\mathbf{O}|\binom{9}{6}=(2^4+1)(2^5-1-1)\binom{9}{6}.$$

Next, fix $S\subset\mathbf{O}$. Then $\dim\langle S\rangle\leqq 6$, so there is some $y\in(\mathbf{P}-\mathbf{O})\cap\langle S\rangle^\perp$. Projecting into $y^\perp/y$ and applying the preceding paragraph, we find that $\langle S\rangle$ has type $\Omega^-(6,2)$. Thus, $|(\mathbf{P}-\mathbf{O})\cap\langle S\rangle^\perp|=2^2+1$, and $N=\binom{17}{6}5$, which is absurd.

Proposition 4.4. *Assume that* $\mathbf{G}(\Sigma)\cong\mathbf{G}(V')$ *for some* V', *and that* V' *has type* $\Omega^+(2m,q)$, $U(2m,q^{1/2})$ *or* $\Omega(2m+1,q)$. *Then* V *contains an ovoid.*

Proof. If $\phi:\mathbf{G}(\Sigma)\to\mathbf{G}(V')$ is an isomorphism, then $\Sigma'=(\Sigma^*)^\phi$ is a spread of V' (cf. (3.3)). Let H be a nonsingular hyperplane of V' (of type $\Omega^-(2m,q)$ if V has type $\Omega(2m+1,q)$). Then $\Sigma''=\{E\cap H \mid E\in\Sigma'\}$ is a spread of H, since each subspace $E\cap H$ has dimension $m-1$. By (3.3ii), $E\cap H=(x^*)^\phi$ for some $x\in\mathbf{P}$. Let $\mathbf{O}$ be the set of $|\Sigma'|$ points x obtained in this manner from H.

We claim that $\mathbf{O}$ is an ovoid. Clearly $|\mathbf{O}|=|\Sigma|=q^{m+\varepsilon}+1$. Assume that x and y are distinct perpendicular points of $\mathbf{O}$. Let $y\in Y<x^\perp$. Then $Y\sim X$ for all $X\in x^*$, so that Y^ϕ is perpendicular to every point of $(x^*)^\phi=E\cap H$. Since $Y^\phi\in(y^*)^\phi<H$, this is impossible. Thus, $\mathbf{O}$ is an ovoid.

Note that the different choices for H produce many ovoids. Conceivably, these can be used to obtain further information.

Corollary 4.5. $\mathbf{G}(\Sigma)\not\cong\mathbf{G}(V')$ *for all* V' *if* V *has type* $\Omega^+(2m,2)$, $m>4$, *or* $\mathrm{Sp}(2m,q)$ *with* q *even.*

Proof. This follows from (4.3) and (4.4), since an $\mathrm{Sp}(2m,q)$ geometry is essentially the same as an $\Omega(2m+1,q)$ geometry when q is even.

Remark 4.6. Suppose that $\mathbf{G}(\Sigma)\cong\mathbf{G}(V')$. If q is odd and V' has type $\Omega(2m+1,q)$, then so does V by (4.3) and (4.4). If q is odd and V' is symplectic, then so is V: if X and Y are nonadjacent in $\mathbf{G}(\Sigma)$ then there are $q-1$ subspaces $Z\neq X,Y$ such that $W\sim X,Y\Rightarrow W\sim Z$; as in §3 this implies that $Z\cap\langle\Sigma(X)\cap Y^\perp,\Sigma(Y)\cap X^\perp\rangle\neq 0$, and hence that V is not orthogonal.

5. Symplectic spreads

Assume that V has type $\mathrm{Sp}(2n, q^e)$, and that Σ' is a spread of V. Set $F = \mathrm{GF}(q^e)$ and $K = \mathrm{GF}(q)$.

If (u, v) is the symplectic form for V, and $T: F - K$ is the trace map, then $T(u, v)$ defines a symplectic form over K, and turns V into an $\mathrm{Sp}(2en, q)$ space. Moreover, Σ' becomes a spread Σ of totally isotropic en-spaces. (Thus, we regard Σ as consisting of K-subspaces and Σ' as consisting of F-subspaces.) If $e \geqq 3$, we can form the strongly regular graphs $\mathbf{G}(\Sigma)$ and $\mathbf{G}(\Sigma')$. These graphs do not even have the same parameters. Thus, extension fields produce large numbers of graphs.

If $X \in \Omega$, let $\Pi(X)$ consist of X and all vertices of $\mathbf{G}(\Sigma)$ joined to X. Then Ω is the set of points, and $\{\Pi(X) \mid X \in \Omega\}$ is the set of blocks, of a symmetric design $\mathbf{D}(\Sigma)$ having the parameters of $\mathrm{PG}(2en - 1, q)$. When $q = 2$, this is a Hadamard design. Clearly, $X \to \Pi(X)$ is a polarity of $\mathbf{D}(\Sigma)$, and X is in $\Pi(X)$. Thus, $\mathbf{D}(\Sigma)$ *is a projective space if and only if* $\mathbf{G}(\Sigma) \cong \mathbf{G}(V)$ (where V is regarded as a K-space).

Recall from (4.4i) that $\mathbf{G}(\Sigma)$ and $\mathbf{G}(V)$ are known to be nonisomorphic if q is even and $en \geqq 3$.

EXAMPLE 5.1. *Desarguesian spreads.* Let $n = 1$ and $e \geqq 3$. If $V = F^2$, then $((\alpha, \beta), (\gamma, \delta)) = \alpha\delta - \beta\gamma$.

The spread Σ' consists of all 1-spaces over F. This produces a spread Σ, a strongly regular graph $\mathbf{G}(\Sigma)$ and a symmetric design $\mathbf{D}(\Sigma)$. We will prove some properties of $\mathbf{G}(\Sigma)$.

(a) The group $G = \mathrm{Sp}(2, q^e) = \mathrm{SL}(2, q^e)$ acts on Σ, and hence on $\mathbf{G}(\Sigma)$. If $M \in \Sigma$ then its stabilizer G_M has order $q^e(q^e - 1)$ and induces a cyclic group of order $q^e - 1$ on the space M. Thus, G is transitive on Ω. If $X \in \Omega$ then G_X fixes every subspace of $\Sigma(X)$. Thus, *the permutation representations of* G *on* $\mathbf{P}$ *and* Ω *are equivalent.*

(b) *If* $X \sim Y$ *and* $\Sigma(X) \neq \Sigma(Y)$, *then* $|L_0(X, Y)| < q + 1$.

PROOF. Let $Z \in L_0(X, Y) - \{X, Y\}$. Then $Z \in XY$ by (3.3). Set $x = X \cap Y^{\perp}$ and $y = Y \cap X^{\perp}$.

The spread Σ can be parametrized by $F \cup \{\infty\}$ as follows: Σ consists of the subspaces

$$\Sigma[\infty] = \{(0, u) \mid u \in F\}; \qquad \Sigma[t] = \{(u, tu) \mid u \in F\} \quad \text{for } t \in F.$$

Since G is 2-transitive on Σ, we may assume that $\Sigma(X) = \Sigma[0]$ and $\Sigma(Y) = \Sigma[\infty]$.

Set $x = \langle(\alpha, 0)\rangle$ and $y = \langle(0,\beta)\rangle$, where $\alpha\beta \neq 0$. Recall that $T: F - K$ was the trace map. Then

$$X = y^\perp \cap \Sigma[0] = \{(u,0) \mid T(u\beta) = 0\},$$

$$Y = x^\perp \cap \Sigma[\infty] = \{(0,u) \mid T(u\alpha) = 0\}.$$

Since $Z \cap \langle x, y\rangle \neq 0$. We may assume that $(\alpha,\beta) \in Z$. Then $\Sigma(Z) = \Sigma[\beta/\alpha]$, so $Z = y^\perp \cap \Sigma[\beta/\alpha] = \{(u, u\beta/\alpha) \mid T(u\beta) = 0\}$.

Similarly, let $\langle(0,\beta')\rangle \in Y - \{y\}$ (so $T(\alpha\beta') = 0$), and form $Z' = \{(u, u\beta'/\alpha) \mid T(u\beta') = 0\}$. Then $Z' \sim Y$ and $Z' < x^\perp$, so $Z' \sim Z$ by (3.2). Set $\langle(\theta, \theta\beta/\alpha)\rangle = Z'^\perp \cap Z$; here $T(\theta\beta) = 0$. Since $(\theta, \theta\beta/\alpha) \in Z'^\perp$, if $T(\mu\beta') = 0$ then $T(\mu\theta\beta\alpha^{-1} - \mu\beta'\alpha^{-1}\theta) = 0$. Thus $\theta\beta\alpha^{-1} - \theta\beta'\alpha^{-1} = k\beta'$ for some $k \in K$.

Consequently, whenever $\beta' \notin K\beta$ and $T(\alpha\beta') = 0$, $T(\beta\alpha\beta'/(\beta - \beta')) = 0$ (since $T(\beta\theta) = 0$). Setting $\gamma = \beta' - \beta$, we see that $T(\alpha\gamma) = 0$ and $\gamma \neq 0$ imply that $T(\alpha^2\beta^2/\alpha\gamma) = T(\alpha\beta(\beta+\gamma)/\gamma) = 0$ (since $T(\alpha\beta) = 0$). Set $\phi = \alpha^2\beta^2$ and $W = \operatorname{Ker} T$. Then $W \cup \{\infty\}$ is invariant under $t \to \phi/t$, as well as $t \to t + w$ for $w \in W$. These permutations generate a subgroup of $\mathrm{PGL}(2, q^e)$ transitive on $W \cup \{\infty\}$. However, $|W \cup \{\infty\}| = q^{e-1} + 1$ does not divide $|\mathrm{PGL}(2, q^e)|$ since $e \geqq 3$. This contradiction completes the proof.

(c) Proposition. $\mathbf{G}(\Sigma) \not\cong \mathbf{G}(V')$ *for all* V', *and* $\operatorname{Aut}\mathbf{G}(\Sigma) = \mathrm{P\Gamma L}(2, q^e)$.

Proof. By (b), (3.3) and (3.4), $\mathbf{G}(\Sigma) \not\cong \mathbf{G}(V')$ for all V', and $\operatorname{Aut}\mathbf{G}(\Sigma) = \operatorname{Aut}\mathbf{G}(V)_\Sigma$. Since $\Gamma\mathrm{L}(2e, q)_\Sigma = \Gamma\mathrm{L}(2, q^e)$ (Dembowski [3, p. 32]), this proves (c).

We now turn to the design $\mathbf{D}(\Sigma)$.

(d) *The partition* Σ^* (cf. (3.2)) *of* Ω *is completely determined by the design* $\mathbf{D}(\Sigma)$. For, if $X \neq Y$ let $l(X, Y)$ denote the intersection of all blocks containing X and Y. If $\Sigma(X) = \Sigma(Y)$ then $l(X, Y) = L(X, Y)$ by (3.3iii). If $\Sigma(X) \neq \Sigma(Y)$ then $|l(X, Y)| < q + 1$: when $X \sim Y$, this follows from (c), while if X and Y are not joined it is proved exactly as in (c). (In fact, $|l(X, Y)| = 2$ whenever $\Sigma(X) \neq \Sigma(Y)$.)

The set $\mathbf{P}$ *can also be recovered from* $\mathbf{D}(\Sigma)$. For, if $M \in \Sigma$ and B is any block, then $B \cap M^* = M^*$ or x^* for some $x \in M$ (by (3.3ii)). This also shows that M^* inherits the structure of a projective space $\mathrm{PG}(e-1, q)$ from $\mathbf{D}(\Sigma)$. In particular, the lines of M can be recovered.

In fact, *all lines of* $\mathrm{PG}(2e-1, q)$ *can be determined*. For, if x and y belong to different members of Σ, then $x^* \cap y^*$ lies in exactly $q + 1$ blocks, and their intersection contains exactly $q + 1$ sets z^*: those z^* such that z is in $\langle x, y\rangle$.

It is now easy to determine the automorphism group of $\mathbf{D}(\Sigma)$; this group does not quite coincide with $\operatorname{Aut}\mathbf{G}(\Sigma)$.

(e) $\operatorname{Aut}\mathbf{D}(\Sigma) \cong \Gamma L(2, q^e)/\mathrm{GF}(q)^*$.

Proof. By (d) $\operatorname{Aut}\mathbf{D}(\Sigma) \leqq P\Gamma L(2e, q)_\Sigma \cong \Gamma L(2, q^e)/\mathrm{GF}(q)^*$. Since $\operatorname{Aut}\mathbf{D}(\Sigma) \geqq \operatorname{Aut}\mathbf{G}(\Sigma)$, we only need to check that the transformation $g:(\alpha,\beta)\to(\alpha\zeta,\beta\zeta)$ of V induces an automorphism of $\mathbf{D}(\Sigma)$ whenever $\zeta \in F^*$.

Let $X < \Sigma[t]$, $t \in F$. Then $X = \Sigma[t] \cap (0,\beta)^\perp = \{(\alpha, t\alpha) \mid T(\alpha\beta) = 0\}$ for some $\beta \neq 0$. Call $X = X(t,\beta)$ in this situation. Note that $X(t,\beta)^g = \{(\alpha, t\alpha) \mid T(\alpha\zeta^{-1}\beta) = 0\} = X(t, \beta\zeta^{-1})$.

Consider $W = \{(0,\beta) \mid T(\beta) = 0\} \in \Omega$. We have

$$\Pi(W) = \Sigma[\infty]^* \cup \{X(t,\beta) \mid t, \beta \in F,\ T(\beta) = 0\},$$

so that

$$\Pi(W)^g = \Sigma[\infty]^* \cup \{X(t,\beta\zeta^{-1}) \mid t, \beta \in F,\ T(\beta) = 0\} = \Pi(W'),$$

where $W' = \{(0,\gamma) \mid T(\gamma\zeta) = 0\}$. Since G centralizes g and is transitive on blocks, it follows that g induces an automorphism, as required.

Remark. The design $\mathbf{D}(\Sigma)$ has a remarkably rich structure. While it seems difficult to determine the size of each intersection of three blocks, it is easy to obtain $|\Pi(X) \cap \Pi(Y) \cap \Pi(Z)|$ whenever $\Sigma(X) = \Sigma(Y)$.

Example 5.2. *Tits spreads.* Let $q^e = 2^{2f+1}$, $e \geqq 3$. The Suzuki group $\mathrm{Sz}(q^e)$ determines a spread Σ' of an $\mathrm{Sp}(4, q^e)$ space (Tits [15]; Lüneburg [11]). This produces a spread Σ of an $\mathrm{Sp}(4e, q)$ space V, as before. By (4.5), $\mathbf{G}(\Sigma) \not\cong \mathbf{G}(V)$. By (5.1c), the strongly regular graphs in (5.1) and (5.2) cannot be isomorphic. By (5.1e), the corresponding symmetric designs are also nonisomorphic.

Remark. Many further examples of symplectic spreads are known (Kantor [9, 10]), but the corresponding graphs do not seem particularly interesting.

6. Orthogonal desarguesian spreads

An $\Omega^\perp(2n+2, q)$ space V can have a spread only if n is odd. If $n \geqq 5$, examples of spreads are known only if q is even (Dillon [5]; Dye [6]; Kantor [9, 10]). These examples are much harder to compute with than the ones in (5.1). Only one type of example, called *desarguesian* in [9], seems halfway tractable computationally. Of course, if $q = 2$ then (4.5) can be applied to any Σ in order to deduce that $\mathbf{G}(\Sigma) \not\cong \mathbf{G}(V)$.

Let q be even, and set $F = \mathrm{GF}(q^n)$, $K = \mathrm{GF}(q)$, $V = K \oplus F \oplus F \oplus K$ and $Q(a, \alpha, \beta, b) = T(\alpha\beta) + ab$, where $T : F \to K$ is the trace map. Then $\Sigma = \{\Sigma[t] \mid t \in F \cup \{\infty\}\}$ is the spread we will study in this section, where (for $t \in F$)

$$\Sigma[\infty] = \{(0, 0, \beta, b) \mid \beta \in F, b \in K\},$$

$$\Sigma[t] = \{(a + T(t\alpha), \alpha, t^2\alpha + ta, T(t\alpha)) \mid \alpha \in F, a \in K\}.$$

There is a group $\mathrm{P\Gamma L}(2, q^n)$ of transformations preserving Q and Σ, fixing $\langle(1, 0, 0, 1)\rangle$, and acting 3-transitively on Σ ([6], [10]).

Proposition 6.1. *If $n \geqq 5$ then* $\mathbf{G}(\Sigma) \not\cong \mathbf{G}(V)$.

We will describe three proofs of this fact. The first involves a reduction to the case $q = 2$: if $\mathbf{G}(\Sigma) \cong \mathbf{G}(V)$, one can use a field automorphism in $\mathrm{P\Gamma L}(2, q^n)$ in order to make this reduction.

The second proof is group theoretic. If $\phi : \mathbf{G}(\Sigma) \to \mathbf{G}(V)$ is an isomorphism and $G = \mathrm{PSL}(2, q^n)$ acts on Σ, then $G^\phi = \phi^{-1}G\phi$ acts on V and preserves the spread Σ^* of (3.3). Moreover, the orbits of G on Ω have lengths $|\Sigma|$ and $|\Omega| - |\Sigma|$, and hence G^ϕ has the same orbit lengths on $\mathbf{P}$. This implies that G^* cannot fix a 1-space of V. But this is impossible by Fong and Seitz [7, (4A)].

The third proof is computational. While it is very unpleasant, it at least has the advantage of producing $\mathrm{Aut}\,\mathbf{G}(\Sigma)$.

Proposition 6.2. *If $n \geqq 5$ then* $\mathrm{Aut}\,\mathbf{G}(\Sigma) = (\mathrm{Aut}\,\mathbf{G}(V))_\Sigma$.

Proof. In view of (3.4), it suffices to prove that *every set $L(X, Y)$ with $X \sim Y$ and $\Sigma(X) \neq \Sigma(Y)$ consists only of X and Y.* Assume that $Z \in L(X, Y) - \{X, Y\}$. By (3.3iv), $Z \in XY$.

Set $x = X \cap Y^\perp$ and $y = X^\perp \cap Y$. Since $\mathrm{P\Gamma L}(2, q^n)$ acts 2-transitively on Σ, we may assume that $\Sigma(X) = \Sigma[0]$ and $\Sigma(Y) = \Sigma[\infty]$. Let $x = \langle(p, \pi, 0, 0)\rangle$ and $y = \langle(0, 0, \beta, b)\rangle$, where $T(\pi\beta) = pb$. We will concentrate on the case $b \neq 0$, since the case $b = 0$ is similar but slightly simpler. We may assume that $(p, \pi, \beta, b) \in Z < \Sigma[s]$ for some $s \in F$. Then $T(s\pi) = b$ and $\beta = s^2\pi + s(p + b)$.

Fix $t \in F - \{0, s\}$. Set $w = \langle(i + T(t\theta), \theta, t^2\theta + ti, T(t\theta))\rangle = \Sigma[t] \cap Z^\perp$ with $i \in K$ and $\theta \in F$. If $pT(s\alpha) + T(\pi(s^2\alpha + sa)) = 0$ then

$$(a + T(s\alpha), \alpha, s^2\alpha + sa, T(s\alpha)) \in \Sigma[s] \cap x^\perp = Z < w^\perp,$$

so that

$$(i + T(t\theta))T(s\alpha) + (a + T(s\alpha))T(t\theta) + T(\theta(s^2\alpha + sa)) + T(\alpha(t^2\theta + ti)) = 0.$$

Since $T(\pi sa) = ab$ and $b \neq 0$, this reduces to $T(\alpha r) = 0$ for all α, where

$$r = (ps + \pi s^2)T(s\theta + t\theta) + b\{i(s+t) + \theta(s^2+t^2)\} = 0. \tag{6.3}$$

Here, $T(s\theta + t\theta) \neq 0$. (For otherwise, $i + \theta(s+t) = 0$ and hence $i + 0 = 0$ and $\theta = 0$.) From $T(\theta r) = 0$ we then deduce $T((ps + \pi s^2)\theta) = b\{i + T(s\theta + t\theta)\}$; thus, i is known in (6.3).

Choose any $\lambda \in F$ such that $\lambda \neq 0$ and $T(\lambda\beta) = T(\lambda t) = T(\lambda t^2\pi) = 0$. Then $(0, \lambda, 0, 0) \in y^\perp \cap \Sigma[\infty] = X$, $(0, 0, \lambda t^2, 0) \in x^\perp \cap \Sigma[0] = Y$, and $(0, \lambda, \lambda t^2, 0) \in \Sigma[t]$. Set $W = (0, \lambda, 0, 0)^\perp \cap \Sigma[t] = (0, \lambda, \lambda t^2, 0)^\perp \cap \Sigma[t]$. Then $W \sim X, Y$, so $W \sim Z$. Thus, $w \in (0, \lambda, 0, 0)^\perp$, so $T(\lambda(t^2\theta + ti)) = 0$ for each choice of λ. Consequently, $t^2\theta + ti \in \langle \beta, t, t^2\pi \rangle$. Elimination of θ in (6.3) produces a nontrivial polynomial of degree $\leqq 4$ over F having t as a root and depending upon p, π, β, b, s and at most four elements of K. Thus, the number of elements t is $q^n - 2 \leqq 4q^5$. This is a contradiction unless $n = 5$ and $q = 2$ or 4; a bit more care eliminates these cases as well.

A very similar argument also handles the case $b = 0$.

Remarks. The preceding technical proof can be summarized as follows. Let $Z \in XY - \{X, Y\}$, $x = X \cap Y^\perp$ and $y = X^\perp \cap Y$. Let $M \in \Sigma - \{\Sigma(X), \Sigma(Y), \Sigma(Z)\}$. Set $w = Z^\perp \cap M$, so $w \in \langle x, y\rangle^\perp$. Let $w \in \langle x', y'\rangle$, where $x' \in X$ and $y' \in Y$. If $v \in M \cap \langle x, y\rangle^\perp$ but $v \notin x'^\perp$, then $v \in \langle x_1, y_1\rangle$ with $x_1 \in X$ and $y_1 \in Y$, and $x_1^\perp \cap M$ is joined to X and Y but not to Z. The proof consisted of showing that $M \cap \langle x, y\rangle^\perp \neq M \cap \langle x, x'\rangle^\perp$ for "most" choices of M.

Presumably, there are better approaches than the preceding computational one. We have not performed similar computations with other $\Omega^+(2n+2, q)$ spreads, nor with any of the known $\Omega^-(2m+2, q)$ spreads.

7. Partial geometries

Throughout this section, V will have type $\Omega^+(2m, 2)$. Let Q denote the corresponding quadratic form, and set $\mathbf{N} = \{\langle v\rangle \mid Q(v) = 1\}$.

Let Σ and Ω be as in §3.

Theorem 7.1. *If $q = 2$ or 3, then $(\mathbf{N}, \Omega, \perp)$ is a partial geometry.*

This result is due to DeClerck, Dye and Thas [2; 14]. The resulting partial geometry will be called pg(Σ).

The point-graph of pg(Σ) is the graph with vertex set $\mathbf{N}$, two different members of $\mathbf{N}$ being joined if and only if they are incident with a member of Ω. The line-graph of pg(Σ) is defined similarly. From (7.1), it is easy to deduce

COROLLARY 7.2. (i) *The point-graph of* $\mathrm{pg}(\Sigma)$ *is* $(\mathbf{N}, \perp)$ *if* $q=2$ *and* $(\mathbf{N}, \not\perp)$ *if* $q=3$.

(ii) *The line-graph of* $\mathrm{pg}(\Sigma)$ *is the graph complementary to* $\mathrm{G}(\Sigma)$.

The point- and line-graphs of a partial geometry are strongly regular. In view of (i), GF(2) and GF(3) are the only fields for which this construction produces a partial geometry. On the other hand, (ii) is the motivation for much of this paper.

From (7.2i) we deduce

COROLLARY 7.3. *Let* Σ *and* Σ' *be spreads of* V. *Then any isomorphism* $\mathrm{pg}(\Sigma) \to \mathrm{pg}(\Sigma')$ *is induced by an orthogonal transformation of* V *sending* Σ *to* Σ'. (*In particular* $\mathrm{Aut}(\mathrm{pg}(\Sigma))$ *is the stabilizer of* Σ *in the projective orthogonal group.*)

Finally, by Patterson [12] and Kantor [9, §§5,10; 10, §9] we have

COROLLARY 7.4. (i) *If* $m=4$ *and* $q=2$ *or* 3, *then* $\mathrm{pg}(\Sigma)$ *is unique up to isomorphism.*

(ii) *If* $m-1$ *is a composite odd integer, then there are at least three nonisomorphic partial geometries* $\mathrm{pg}(\Sigma)$ *arising from an* $\Omega^+(2m,2)$ *space, all having the same parameters.*

No examples with $m>4$ and $q=3$ are presently known.

8. Partial geometries in $\Omega^+(8,2)$ space

Haemers and van Lint [8] constructed a partial geometry $\mathrm{pg}(HvL)$ as follows. Take Z_2^9, and define a quadratic form Q as in the proof of (4.3). Let V denote the set of vectors of even weight. Then V becomes an $\Omega^+(8,2)$ space. The group $G=\mathrm{P\Gamma L}(2,8)$ acts on this space by permuting coordinates. Haemers and van Lint construct 120 cocliques of size 9, lying in G-orbits of sizes 1, 63 and 56. Then $\mathrm{pg}(HvL)$ has $\mathbf{P}$ as point set and these 120 cocliques as line set.

THEOREM 8.1. $\mathrm{pg}(HvL)$ *is isomorphic to the dual of* $\mathrm{pg}(\Sigma)$, *where* Σ *is a spread of* V.

This can be proved either using triality or coset geometries. We will use the latter approach.

If Σ is a spread in V, then it is preserved by a group of orthogonal transformations isomorphic to A_9. The stabilizer of a nonsingular point b of V gives us a group $G=\mathrm{P\Gamma L}(2,8)$ preserving Σ (compare Dye [6]; Kantor [9]; and §6). Moreover, G is 3-transitive on Σ, and has exactly 3 orbits on the set $\mathbf{N}$ defined in §7 (namely, $\{b\}$, $b^\perp \cap \mathbf{N}-\{b\}$ of size 63 and $\mathbf{N}-b^\perp$ of size 56) and 2 on

Ω (namely, $\Sigma_b = \{b^\perp \cap M \mid M \in \Sigma\}$ and $\Omega - \Sigma_b$). If $c \in \mathbf{N} - \{b\}$, then $|G_c| = 24$ or 27, and G_c has an orbit on Σ_c of length 8 or 9, respectively. Thus, G acts on the dual of pg(Σ) so as to satisfy the following conditions:

(i) G fixes a line L, and induces PΓL(2, 8) on it.

(ii) G is transitive on the points not on L; the stabilizer of such a point p has order 12.

(iii) G_p has two orbits of lines on p: those meeting L and those not.

(iv) If L' is a line other than L, then $|G_{L'}| = 24$ or 27.

By construction, pg(HvL) has the same properties.

Let $A < G$ with $|A| = 12$; then $A \cong A_4$ and all such subgroups are conjugate (since the centralizer of an involution in PΓL(2, 8) is $Z_2 \times A_4$). Let $S, T < G$ with $|S| = 24$ and $|T| = 27$. Set $L = \mathrm{PG}(1, 8)$. Define a geometry as follows (where $g \in G$).

Points: points x of L; cosets Ag.

Lines: L; cosets Sg; cosets Tg.

Incidence: xIL; $xISg \Leftrightarrow S$ fixes $x^{g^{-1}}$; $AgISg' \Leftrightarrow Ag \cap Sg' \neq \varnothing$; and $AgITg' \Leftrightarrow Ag \cap Tg' \neq \varnothing$.

In view of the flag-transitivity implicit in (iii), Dembowski [3, p. 15] implies that both pg(Σ) and pg(HvL) are isomorphic to the above coset geometry.

REMARKS. (1) If Σ is the desarguesian spread of an $\Omega^+(2n+2, 2)$ space, then pg(Σ) can be described as a coset geometry in a very similar manner. In this more general case, $G = \mathrm{P\Gamma L}(2, 2^n)$, $|A| = 2^{n-1}n$, $|S| = 2^n n$, $|T| = (2^n + 1)n$ and $L = \mathrm{PG}(1, 2^n)$.

(2) The dual of the partial geometry constructed by Cohen [1] is undoubtedly also isomorphic to pg(Σ). That partial geometry has the correct point-graph (cf. (7.2i)), and Aut pg(Σ) has a subgroup A_5 which seems to behave like the A_5 in [1]. However, the complicated construction [1] makes the required identification difficult.

9. Some designs

The partial geometries in §7 can be described as $(\Omega, \mathbf{N}^\perp, \subset)$, where $\mathbf{N}^\perp = \{b^\perp \mid b \in \mathbf{N}\}$ is a suitable set of hyperplanes. In this section, we will present a similar construction for some less interesting objects.

Let V be an $\Omega(2m+1, 2)$ space, and set $q = 2^m$. Let Σ and Ω be as usual, and let $\mathbf{N}^-$ denote the set of all hyperplanes of type $\Omega^-(2m, 2)$.

PROPOSITION 9.1. *$(\mathbf{N}^-, \Omega, \supset)$ is a design with parameters $v = \frac{1}{2}q(q-1)$, $k =$*

$\frac{1}{2}q$, $b = q^2 - 1$, $r = q + 1$ and $\lambda = 1$. Its line graph is the graph complementary to **G(Σ)**.

PROOF. The values of v, k, b, and r are easy to compute. If X and Y are distinct, nonadjacent members of Ω, then $\langle X, Y\rangle$ has type $\Omega^+(2m-2,2)$, so that $\langle X, Y\rangle^\perp$ has type $\Omega(3,2)$, and hence has exactly one anisotropic 2-space. Thus, any two distinct members of N^- have in common at most one element of Ω. A standard counting argument completes the proof.

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